ELSEVIER

Journal of Pure and Applied Algebra 111 (1996) 169–180

JOURNAL OF PURE AND APPLIED ALGEBRA

Inclusion of poset homology into Lie algebra homology

Iztok Hozo

Department of Mathematics, Indiana University Northwest, 3400 Broadway, Gary, IN 46408, USA

Communicated by C.A. Weibel; received 13 March 1995

Abstract

Given a poset P on the set $\{1,2,\dots,n\}$, we define the nilpotent Lie algebra L_P to be the span of all elementary matrices $z_{i,j}$, such that i is less than j in P. We show that for a particular class of partially ordered sets the homology of poset is included in the homology of the corresponding Lie algebra. A necessary condition is that the poset P has both the minimum $\hat{0}$ and the maximum $\hat{1}$.

1991 Math. Subj. Class.: 17B56, 05E25

1. Preliminary results

Suppose that P is a finite poset with $\hat{0}$ and $\hat{1}$. We define an *r-chain* to be a chain of length $(r+1)$ from $\hat{0}$ to $\hat{1}$. The zero-chain, is the chain $\hat{0} < \hat{1}$.

Define the set $\mathscr{C}_r(P)$ to be the set of 0–1 chains of length r in the poset P. By abuse of notation we will use the same name for the complex vector space C_r or $\mathscr{C}_r(P)$, with basis the set of r-chains. The C_r's are called *chain spaces*. The map $\partial_r : C_r \to C_{r-1}$, called the *boundary map*, is defined by

$$\partial_r(\hat{0} < x_1 < \cdots < x_r < \hat{1}) = \sum_{i=1}^{r}(-1)^{i-1}(\hat{0} < x_1 < \cdots < \widehat{x_i} < \cdots < x_r < \hat{1})$$

and has the property that $\partial_{r-1} \circ \partial_r = 0$. The *homology of a poset* is thus defined to be:

$$H_r(P) = \mathrm{Ker}(\partial_r)/\mathrm{Im}(\partial_{r+1}).$$

Definition 1. A *standard labeling* of the poset P is a total ordering of the elements of P such that whenever $x <_P y$, x also precedes y in that total ordering.

SSDI 0022-4049(95)00121-2

Since P is a partial order, i.e. transitive, there always is such a labeling. Fix a standard labeling of the poset P.

We define a Lie algebra L_P corresponding to the poset P in the following way. First, for every relation $x <_P y$ in the poset P, i.e. for every two elements $x, y \in P$ such that $x <_P y$ we can define the matrix $z_{x,y}$ having all entries equal to zero, except for exactly one entry equal to 1, namely the entry at the position x, y in the standard labeling of the poset P.

All matrices $z_{x,y}$ are strictly upper triangular because of our labeling. So L_P is a subalgebra of the algebra of strictly upper triangular matrices $T_n(\mathbb{C})$. The Lie algebras L_P obtained from distinct labelings are isomorphic – the labeling only specifies an embedding of L_P in the $n \times n$ matrices.

The Lie algebra homology of L_P is defined to be the homology of the chain complex whose kth graded piece is $\Gamma_k(L_P) = \bigwedge^k(L_P)$ where the exterior power is over the field of complex numbers $\mathbb{C}$.

The boundary operator is defined in the following way:

$$\begin{aligned}&\partial_k(z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_k,y_k})\\ &\quad= \sum_{i<j}(-1)^{i+j-1}([z_{x_i,y_i}, z_{x_j,y_j}] \wedge z_{x_1,y_1} \wedge \cdots \wedge \widehat{z_{x_i,y_i}} \wedge \cdots \wedge \widehat{z_{x_j,y_j}} \wedge \cdots \wedge z_{x_k,y_k}).\end{aligned}$$

Abusing the notation we will call this boundary operator also ∂, leaving to the reader to determine in which spaces we are actually residing at the moment, i.e. whether $\partial : C_k(P) \to C_{k-1}(P)$ or $\partial : \Gamma_k(L_P) \to \Gamma_{k-1}(L_P)$. It will always be clear from the text which map we are talking about and will greatly improve the readability of this paper.

Later in this work we will talk about an operator, called the Laplacian of a complex, for which we need to identify the transpose of the boundary maps.

In this case – the case of the poset homology – the transpose of the boundary map is not so difficult to evaluate. We are in fact transposing the matrix of the boundary map with respect to the basis of r-chains.

Lemma 1. *The transpose of the boundary operator $\partial : C_k(P) \to C_{k-1}(P)$ (viewed as a linear map), is given by the following expression:*

$$\begin{aligned}&\partial^t(\hat{0} < x_1 < \cdots < x_r < \hat{1})\\ &\quad= \sum_{i=0}^{r} \sum_{x_i < y < x_{i+1}} (-1)^i(\hat{0} < x_1 < \cdots < x_i < y < x_{i+1} < \cdots < x_r < \hat{1})\end{aligned}$$

where $x_0 = \hat{0}$ and $x_{r+1} = \hat{1}$.

Proof. All we need to prove is that

$$\begin{aligned}&\partial(\hat{0} < x_1 < \cdots < x_i < y < x_{i+1} \cdots < x_r < \hat{1})\\ &\quad= \cdots + (-1)^i(\hat{0} < x_1 < \cdots < x_r < \hat{1}) + \cdots,\end{aligned}$$

i.e. we need to prove that the chain $(\hat{0} < x_1 < \cdots < x_r < \hat{1})$ appears with the coefficient $(-1)^i$ in the expansion of $\partial(\hat{0} < x_1 < \cdots < x_i < y < x_{i+1} \ldots < x_r < \hat{1})$ with respect to our basis of r-chains.

This is clear by inspection of the definition of ∂:

$$\partial(\hat{0} < x_1 < \cdots < x_r < \hat{1})$$
$$= \sum_j (-1)^{j-1}(\hat{0} < x_1 < \cdots < \widehat{x_j} < \cdots < x_r < \hat{1})$$

and in particular, if we set $j = i + 1$ we have

$$\partial(\hat{0} < x_1 < \cdots < x_i < y < x_{i+1} < \cdots < x_r < \hat{1})$$
$$= \cdots + (-1)^i(\hat{0} < x_1 < \cdots < x_r < \hat{1}) + \cdots$$

This completes the proof of the lemma. □

For the Lie algebra homology, the boundary map transpose needs some additional setup.

We will now define an orthogonal inner product $\langle \cdot, \cdot \rangle$ on the product $\oplus \Gamma_r$, such that $\langle \Gamma_r, \Gamma_s \rangle = 0$ whenever $r \neq s$. We will restrict our argument to the subspaces of the nilpotent Lie algebra $T_n(\mathbb{C})$, of all strictly upper triangular matrices, with standard basis $\{z_{i,j} : 1 \le i < j \le n\}$, so we can define this product naturally:

Definition 2. Let L be a Lie algebra, $L \subset T_n(\mathbb{C})$. Define an inner product for standard basis elements $v, w \in L$ by

$$\langle v, w \rangle = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{otherwise}, \\ 0 & \text{if } v \text{ and } w \text{ have different exterior degrees.} \end{cases}$$

Extend this to the exterior algebra, i.e. to the complexes mentioned above.

Definition 3. Suppose that $v = v_1 \wedge \cdots \wedge v_k$ and $w = w_1 \wedge \cdots \wedge w_k$. Then define the inner product:

$$\langle v, w \rangle = \det(\langle v_i, w_j \rangle)_{1 \le i,j \le k}.$$

Note that this can be written also as

$$\langle v, w \rangle = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_i \langle v_i, w_{\sigma(i)} \rangle = \begin{cases} \operatorname{sgn}(\sigma) & \text{iff } v_i = w_{\sigma(i)} \text{ for all } i, \\ 0 & \text{otherwise}. \end{cases}$$

In other words, the product of two pure wedges of basis elements is nonzero if and only if two pure wedges differ only in the order of the elements, and in that case, the product is just the sign of the permutation that changes one into another.

Define δ_r mapping Γ_r into Γ_{r+1} by

$$\langle \delta_r v, w\rangle = \langle v, \partial_{r+1} w\rangle$$

over all $v \in \Gamma_r$, and all $w \in \Gamma_{r+1}$. It is enough to calculate δ on pure wedges (as in our definitions), since the inner product and δ are both linear functions.

In this case we claim that the map δ is nothing else but transpose of the boundary map with respect to the obvious basis of the pure wedges.

First we will show that δ is given by the following formula:

Lemma 2. *If δ is a function $\delta : \Gamma_r \to \Gamma_{r+1}$, defined by*

$$\langle \delta_r v, w\rangle = \langle v, \partial_{r+1} w\rangle$$

then

$$\delta_r(z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_r,y_r})$$
$$= \sum_{s=1}^{r} (-1)^{s-1} \sum_{x_s<l<y_s} z_{x_1,y_1} \wedge \cdots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \cdots \wedge z_{x_r,y_r}.$$

Proof. The proof will be by induction. If $r = 1$, we want to prove that

$$\delta_1(z_{x,y}) = \sum_{x<l<y} z_{x,l} \wedge z_{l,y}.$$

By definition

$$\langle \delta_1(z_{x,y}), z_{x_1,y_1} \wedge z_{x_2,y_2}\rangle = \langle z_{x,y}, [z_{x_1,y_1}, z_{x_2,y_2}]\rangle$$
$$= \begin{cases} 1 & \text{if } x = x_1 < y_1 = x_2 < y_2 = y, \\ -1 & \text{if } x = x_2 < y_2 = x_1 < y_1 = y, \\ 0 & \text{otherwise.} \end{cases}$$

Thus for any l, such that $x < l < y$, the product of $\delta_1(z_{x,y})$ and $z_{x,l} \wedge z_{l,y}$ is 1. This means that

$$\delta_1(z_{x,y}) = \sum_{x<l<y} z_{x,l} \wedge z_{l,y}.$$

To prove the general case it is enough to establish following connection between δ_r and δ_1:

Proposition 3. *For $v = (v_1 \wedge \cdots \wedge v_r) \in \Gamma_r$, δ_r satisfies*

$$\delta_r(v_1 \wedge \cdots \wedge v_r) = \sum_s (-1)^{s-1} v_1 \wedge \cdots \wedge \delta_1(v_s) \wedge \cdots \wedge v_r$$

Together with the calculation of δ_1 that we just completed, this proposition will prove the lemma.

Proof Proposition 3.

$$\langle \delta v, w\rangle = \langle v, \partial w\rangle$$

$$= \sum_{i<j}(-1)^{i+j-1}\langle v_1 \wedge \cdots \wedge v_r,$$

$$[w_i, w_j] \wedge w_1 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_{r+1}\rangle \tag{1}$$

$$= \sum_{i<j}(-1)^{i+j-1}\langle v_1[w_i, w_j]\rangle$$

$$\cdot\langle v_2 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_{r+1}\rangle \tag{2}$$

$$- \sum_{l=1}^{r+1}\sum_{i<j}\varepsilon(i,j;l)\langle v_1, w_l\rangle$$

$$\cdot\langle v_2 \wedge \cdots \wedge v_r, [w_i, w_j] \wedge w_1 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots \wedge \widehat{w}_l \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_{r+1}\rangle \tag{3}$$

where

$$\varepsilon(i,j;l) = \begin{cases} (-1)^{i+j+l}, & l < i < j, \\ (-1)^{i+j+l-1}, & i < l < j, \\ (-1)^{i+j+l}, & i < j < l. \end{cases}$$

First deal with the summation (2):

$$\sum_{i<j}(-1)^{i+j-1}\langle v_1, [w_i, w_j]\rangle$$

$$\cdot\langle v_2 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_{r+1}\rangle$$

$$= \sum_{i<j}(-1)^{i+j-1}\langle \delta_1 v_1, w_i \wedge w_j\rangle$$

$$\cdot\langle v_2 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge \widehat{w}_i \wedge \cdots \wedge \widehat{w}_j \wedge \cdots \wedge w_{r+1}\rangle$$

$$= \langle \delta_1(v_1) \wedge v_2 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_{r+1}\rangle.$$

Now the summation (3):

$$\sum_{l=1}^{r+1}\sum_{i<j}\varepsilon(i,j;l)\langle v_1, w_l\rangle$$

$$\cdot\langle v_2\wedge\cdots\wedge v_r, [w_i,w_j]\wedge w_1\wedge\cdots\wedge\widehat{w}_i\wedge\cdots\wedge\widehat{w}_l\wedge\cdots\wedge\widehat{w}_j\wedge\cdots\wedge w_{r+1}\rangle$$

$$=\sum_l(-1)^{l-1}\langle v_1,w_l\rangle\cdot\begin{cases}(-1)^{i+j-1}, & l<i<j\\ (-1)^{i+j}, & i<l<j\\ (-1)^{i+j-1}, & i<j<l\end{cases}$$

$$\cdot\langle v_2\wedge\cdots\wedge v_r, [w_i,w_j]\wedge w_1\wedge\cdots\wedge\widehat{w}_i\wedge\cdots\wedge\widehat{w}_j\wedge\cdots\wedge\widehat{w}_l\wedge\cdots\wedge w_{r+1}\rangle$$

$$=\sum_{l=1}^{r+1}(-1)^{l-1}\langle v_1,w_l\rangle\cdot\langle v_2\wedge\cdots\wedge v_r, \partial(w_1\wedge\cdots\wedge\widehat{w}_l\wedge\cdots\wedge w_{r+1}\rangle$$

$$=\sum_{l=1}^{r+1}(-1)^{l-1}\langle v_1,w_l\rangle\cdot\langle\delta(v_2\wedge\cdots\wedge v_r), w_1\wedge\cdots\wedge\widehat{w}_l\wedge\cdots\wedge w_{r+1}\rangle$$

$$=\langle v_1\wedge\delta(v_2\wedge\cdots\wedge v_r), w_1\wedge\cdots\wedge\widehat{w}_l\wedge\cdots\wedge w_{r+1}\rangle.$$

This proves the following relation:

$$\delta(v_1\wedge\cdots\wedge v_r)=(\delta(v_1)\wedge v_2\wedge\cdots\wedge v_r)-(v_1\wedge\delta(v_2\wedge\cdots\wedge v_r))$$

which by induction means that

$$\delta(v_1\wedge\cdots\wedge v_r)=\sum_s(-1)^{s-1}v_1\wedge\cdots\wedge\delta_1(v_s)\wedge\cdots\wedge v_r,$$

i.e. proves Proposition 3. □

This completes the proof of Lemma 2. □

Remark. It is easy to check that $\delta_{r+1}\delta_r=0$, thus δ_* defines a coboundary operator, and so we can define cohomology to be

$$H^r(L)=\mathrm{Ker}(\delta_r)/\mathrm{Im}(\delta_{r-1}).$$

We claim that the transformation δ described by this lemma is nothing but transpose of the boundary map with respect to the obvious basis for complex Γ_r, i.e. with respect to the basis $\{z_{x_1,y_1}\wedge\cdots\wedge z_{x_r,y_r}\}$.

But to prove that, it is enough to show that the coefficient of the pure wedge $z_{x_1,y_1}\wedge z_{x_2,y_2}\wedge\cdots\wedge z_{x_r,y_r}$ in $\partial(z_{x_1,y_1}\wedge\cdots\wedge z_{x_s,l}\wedge z_{l,y_s}\wedge\cdots\wedge z_{x_r,y_r})$ for any $l\in(x_s,y_s)$ is $(-1)^{s-1}$, i.e.

$$\partial(z_{x_1,y_1}\wedge\cdots\wedge z_{x_s,l}\wedge z_{l,y_s}\wedge\cdots\wedge z_{x_r,y_r})$$
$$=\cdots+(-1)^{s-1}(z_{x_1,y_1}\wedge z_{x_2,y_2}\wedge\cdots\wedge z_{x_r,y_r})+\cdots$$

and this is not difficult, since by the definition of ∂:

$$\begin{aligned}
&\partial(z_{x_1,y_1} \wedge \cdots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \cdots \wedge z_{x_r,y_r}) \\
&= \sum_{p<q} (-1)^{p+q-1} [z_{x_p,y_p}, z_{x_q,y_q}] \wedge z_{x_1,y_1} \wedge \cdots \wedge \widehat{z_{x_p,y_p}} \wedge \cdots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \cdots \\
&\quad \ldots \wedge \widehat{z_{x_q,y_q}} \wedge \cdots \wedge z_{x_r,y_r}.
\end{aligned}$$

If we set $i = s$ and $j = s + 1$ we have

$$\begin{aligned}
&\partial(z_{x_1,y_1} \wedge \cdots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \cdots \wedge z_{x_r,y_r}) \\
&= \cdots + (-1)^{s+(s+1)-1} [z_{x_s,l}, z_{l,y_s}] \wedge z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \\
&\quad \cdots \wedge \widehat{z_{x_s,y_s}} \wedge \cdots \wedge z_{x_r,y_r} + \cdots
\end{aligned}$$

but $[z_{x_s,l}, z_{l,y_s}] = z_{x_s,y_s}$ and we can return that element to the sth position by doing $s-1$ transpositions. All taken in account, we get

$$\begin{aligned}
&\partial(z_{x_1,y_1} \wedge \cdots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \cdots \wedge z_{x_r,y_r}) \\
&= \cdots + [z_{x_s,l}, z_{l,y_s}] \wedge z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge \widehat{z_{x_s,y_s}} \wedge \cdots \wedge z_{x_r,y_r} + \cdots \\
&= \cdots + (-1)^{s+1} z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_s,y_s} \wedge \cdots \wedge z_{x_r,y_r} + \cdots.
\end{aligned}$$

We need to check that this is the only possibility to obtain our pure wedge $\zeta = z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_r,y_r}$. Note that ζ cannot contain either $z_{x_s,l}$ or z_{l,y_s} as a constituent since those two elements were erased from the source.

Suppose ζ appears more than once. Our ζ has to appear as a summand in $\partial\mu$ for some μ – hence two of the intervals from the source were glued together, i.e. in the result ζ we have had some t and m, such that

$$\partial(z_{x_1,y_1} \wedge \cdots \wedge z_{x_t,m} \wedge \cdots \wedge z_{m,y_t} \wedge \cdots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \cdots \wedge z_{x_r,y_r}) = \cdots + \zeta + \cdots$$

But the result of shrinking $z_{x_t,m}$ and z_{m,y_t} into one interval z_{x_t,y_t} does not alter both of $z_{x_s,l}$ and z_{l,y_s}. It could alter one of them, but not both. Thus one of the $z_{x_s,l}$ and z_{l,y_s} survives in ζ and this is impossible, as we noted above.

Note that we can change the order of the elements in the pure wedges, and obtain a slightly different form for δ:

$$\begin{aligned}
&\delta_r(z_{x_1,y_1} \wedge z_{x_2,y_2} \wedge \cdots \wedge z_{x_r,y_r}) \\
&= \sum_{s=1}^{r} (-1)^{s-1} \sum_{x_s<l<y_s} z_{x_1,y_1} \wedge \cdots \wedge z_{x_s,l} \wedge z_{l,y_s} \wedge \cdots \wedge z_{x_r,y_r}.
\end{aligned}$$

This is the form for the $\delta = \partial^{\mathrm{t}}$ we will use.

The transposes will be simultaneously denoted by ∂^{t}, where $\partial^{\mathrm{t}} : C_k(P) \to C_{k+1}(P)$ or $\partial^{\mathrm{t}} : \Gamma_k(L_P) \to \Gamma_{k+1}(L_P)$.

Definition 4. Define the *Laplacian operator* $L_r : \Gamma_r \to \Gamma_r$ by

$$L_r = \delta_{r-1}\partial_r + \partial_{r+1}\delta_r.$$

Theorem 1 (Kostant [5]). *Let* $B = \{\beta_1, \ldots, \beta_d\}$ *be a basis for* $Ker(L_r)$. *Then* B *is simultaneously a complete set of representatives of* $H^r(L)$ *and* $H_r(L)$. *In particular* $\dim(H^r(L)) = \dim(H_r(L)) = \dim(Ker(L_r))$.

2. Insertion map

Now define the following mapping:

Definition 5. The *insertion map* ψ mapping $C_k(P) \to \Gamma_{k+1}(L_P)$ is defined by:

$$\psi(\hat{0} < x_1 < \cdots < x_k < \hat{1}) = (z_{\hat{0},x_1} \wedge z_{x_1,x_2} \wedge z_{x_2,x_3} \wedge \cdots \wedge z_{x_k,\hat{1}}).$$

We immediately have these two simple lemmas:

Lemma 4. *Let* P *be a poset, with both* $\hat{0}$ *and* $\hat{1}$. *Let* L_P *be the corresponding Lie algebra. Then*

$$\partial\psi = \psi\partial.$$

Proof. This can be proved directly starting with the expression on the left and ending with the expression on the right in the above claim. In particular,

$$\begin{aligned}
&\partial\psi(\hat{0} < x_1 < \cdots < x_k < \hat{1}) \\
&\quad = \partial(z_{\hat{0},x_1} \wedge z_{x_1,x_2} \wedge \cdots \wedge z_{x_k,\hat{1}}) \\
&\quad = \sum_{i<j} (-1)^{i+j-1}([z_{x_i,x_{i+1}}, z_{x_j,x_{j+1}}] \wedge z_{\hat{0},x_1} \wedge \cdots \wedge z_{x_k,\hat{1}}) \\
&\qquad \text{(the bracket above is nonzero only when } i+1=j) \\
&\quad = \sum_{i} (-1)^{2i}(z_{x_i,x_{i+2}} \wedge z_{\hat{0},x_1} \wedge \cdots \wedge z_{x_k,\hat{1}}) \\
&\qquad \text{(put first element back at the } i\text{th place)} \\
&\quad = \sum_{i} (-1)^{i}(z_{\hat{0},x_1} \wedge \cdots \wedge z_{x_i,x_{i+2}} \wedge \cdots \wedge z_{x_k,\hat{1}}) \\
&\quad = \sum_{i} (-1)^{i}\psi(\hat{0} < x_1 < \cdots < x_i < \widehat{x}_{i+1} < x_{i+2} < \cdots < x_k < \hat{1}) \\
&\quad = \psi\left(\sum_{i} (-1)^{i}(\hat{0} < x_1 < \cdots < x_i < \widehat{x}_{i+1} < x_{i+2} < \cdots < x_k < \hat{1})\right) \\
&\quad = \psi\partial(\hat{0} < x_1 < \cdots < x_k < \hat{1})
\end{aligned}$$

Thus $\psi\partial = \partial\psi$. $\square$

Lemma 5. *Let P be a poset, with both $\hat{0}$ and $\hat{1}$. Let L_P be the corresponding Lie algebra. Let ∂^t be the transpose of the map ∂, in both cases, the poset homology case and in the Lie algebra homology case. Then*

$$\partial^t \psi = \psi \partial^t.$$

Proof. Carefully applying the definitions, we get

$$\begin{aligned}
&\partial^t \psi(\hat{0} < x_1 < \cdots < x_k < \hat{1}) \\
&= \partial^t (z_{\hat{0},x_1} \wedge z_{x_1,x_2} \wedge \cdots \wedge z_{x_k,\hat{1}}) \\
&= \sum_i \sum_{x_i < y < x_{i+1}} (-1)^i (z_{\hat{0},x_1} \wedge \cdots \wedge z_{x_i,y} \wedge z_{y,x_{i+1}} \wedge \cdots \wedge z_{x_k,\hat{1}}) \\
&= \sum_i (-1)^i \sum_{x_i < y < x_{i+1}} \psi(\hat{0} < x_1 < \cdots < x_i < y < x_{i+1} < \cdots < x_k < \hat{1}) \\
&= \psi \left(\sum_i (-1)^i \sum_{x_i < y < x_{i+1}} (\hat{0} < x_1 < \cdots < x_i < y < x_{i+1} < \cdots < x_k < \hat{1}) \right) \\
&= \psi \partial^t (\hat{0} < x_1 < \cdots < x_k < \hat{1}).
\end{aligned}$$

Thus $\psi \partial^t = \partial^t \psi$. $\square$

This permits us to quickly prove the following theorem:

Theorem 2. *Let P be a poset, with both $\hat{0}$ and $\hat{1}$. Let L_P be the corresponding Lie algebra. The Laplacian $\mathscr{L}$ commutes with the Insertion map, i.e.*

$$\mathscr{L}_{L_P} \psi = \psi \mathscr{L}_P.$$

Proof. Recall that in both cases, the poset homology and Lie algebra homology, the Laplacian $\mathscr{L}$, is defined by $\mathscr{L} = \partial \partial^t + \partial^t \partial$. All we need to do now, is to apply the previous two lemmas. Doing so we obtain

$$\mathscr{L}\psi = (\partial \partial^t \psi + \partial^t \partial \psi) = (\partial \psi \partial^t + \partial^t \psi \partial) = (\psi \partial \partial^t + \psi \partial^t \partial) = \psi \mathscr{L}$$

which is what we wanted to prove. $\square$

From Theorem 1 above on Laplacians [5], we know that any basis for the null-space of the Laplacian is a complete set of representatives for a basis of the homology of the corresponding chain complex, i.e. $\mathrm{Ker}(\mathscr{L}_k) \cong H(C_k)$.

Theorem 3 (Insertion theorem). *Let P be a poset, with both $\hat{0}$ and $\hat{1}$. Let $H_*(P)$ be the homology of the poset P, and let $H_*(L_P)$ be the homology of the corresponding Lie algebra L_P. Then for every k, there exists a subspace K, $K \subset H_{k+1}(L_P)$, such that $H_k(P) \cong K$.*

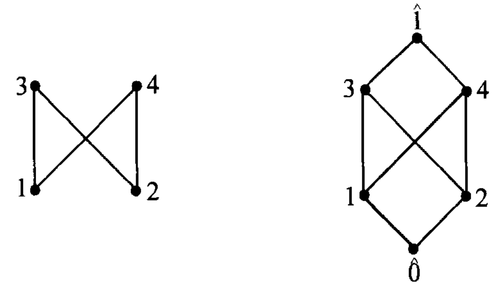

Fig. 1. Example: poset P and $\hat{P}$.

Proof. First note that the insertion map is a good candidate for the isomorphism needed above between a set K and the homology $H_k(P)$. In other words we claim that $K = \psi(H_k(P))$. Note that ψ does map $H_k(P)$ into $H_{k+1}(L_P)$.

According to the result mentioned above, a representative x of the poset homology $H_k(P)$ can be chosen from the kernel of the Laplacian, i.e., $\mathscr{L}_P(x) = 0$. Observe $\mathscr{L}_{L_P}(\psi(x))$.

$$\mathscr{L}_{L_P}(\psi(x)) = \psi(\mathscr{L}_P(x)) = \psi(0) = 0.$$

Thus, all we need for the completion of the proof is the fact that ψ is injective. But that is obvious from the definition of ψ.

Therefore the $(k+1)$st homology of the Lie algebra corresponding to the poset P contains the kth homology of the poset. $\square$

2.1. Example of the insertion

Suppose that the poset P is given in Fig. 1.

For the sake of simplicity we will observe just one of the homologies, say $H_2(P)$ embedded into the Lie algebra Homology $H_3(L_P)$.

First we have to evaluate the second homology of the poset P, $H_2(P) = \operatorname{Ker} \partial_2 / \operatorname{Im} \partial_3$. Note that $\partial_3 = 0$, so the second homology is the kernel of the boundary map ∂_2.

$$\partial_2 : C_2(P) \to C_1(P),$$

$$\partial_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where the matrix above is written with respect to the basis

$$\{(\hat{0} < 1 < 3 < \hat{1}), (\hat{0} < 1 < 4 < \hat{1}), (\hat{0} < 2 < 3 < \hat{1}), (\hat{0} < 2 < 4 < \hat{1})\}$$

of $C_2(P)$ and the basis

$$\{(\hat{0} < 1 < \hat{1}), (\hat{0} < 2 < \hat{1}), (\hat{0} < 3 < \hat{1}), (\hat{0} < 4 < \hat{1})\}$$

of $C_1(P)$.

Thus the map ∂_2 has a kernel of dimension 1, or in other words, $H_2(P) = \mathbb{C}$. The basis element of the second homology is

$$\begin{aligned} x = &(\hat{0} < 1 < 3 < \hat{1}) - (\hat{0} < 1 < 4 < \hat{1}) \\ &-(\hat{0} < 2 < 3 < \hat{1}) + (\hat{0} < 2 < 4 < \hat{1}). \end{aligned}$$

According to our Insertion theorem $\psi(x)$ will be in the homology of the corresponding Lie algebra, in fact in the third homology:

$$\begin{aligned} y = \psi(x) = &(z_{\hat{0},1} \wedge z_{1,3} \wedge z_{3,\hat{1}}) - (z_{\hat{0},1} \wedge z_{1,4} \wedge z_{4,\hat{1}}) \\ &-(z_{\hat{0},2} \wedge z_{2,3} \wedge z_{3,\hat{1}}) + (z_{\hat{0},2} \wedge z_{2,4} \wedge z_{4,\hat{1}}) \end{aligned}$$

If we evaluate $\partial_3(y)$, we obtain

$$\begin{aligned} \partial_3(y) &= \partial_3(z_{\hat{0},1} \wedge z_{1,3} \wedge z_{3,\hat{1}}) - \partial_3(z_{\hat{0},1} \wedge z_{1,4} \wedge z_{4,\hat{1}}) \\ &\quad -\partial_3(z_{\hat{0},2} \wedge z_{2,3} \wedge z_{3,\hat{1}}) + \partial_3(z_{\hat{0},2} \wedge z_{2,4} \wedge z_{4,\hat{1}}) \\ &= (z_{\hat{0},3} \wedge z_{3,\hat{1}}) + (z_{\hat{0},1} \wedge z_{1,\hat{1}}) \\ &\quad -((z_{\hat{0},4} \wedge z_{4,\hat{1}}) + (z_{\hat{0},1} \wedge z_{1,\hat{1}})) \\ &\quad -((z_{\hat{0},3} \wedge z_{3,\hat{1}}) + (z_{\hat{0},2} \wedge z_{2,\hat{1}})) \\ &\quad +(z_{\hat{0},4} \wedge z_{4,\hat{1}}) + (z_{\hat{0},2} \wedge z_{2,\hat{1}}) = 0. \end{aligned}$$

So $\psi(x)$ is in the kernel of the boundary ∂_3. Consider the fourth boundary operator for the Lie algebra homology in this example. The fourth chain space $\Gamma_4(L_P)$ is one-dimensional, spanned by a single vector $v = z_{1,3} \wedge z_{1,4} \wedge z_{2,3} \wedge z_{2,4}$ and the image of this element $\partial_4(v)$ is zero, since no two matrices in v can bracket to a nonzero result. Thus $\psi(x) \in \mathrm{Ker}(d_3)/\mathrm{Im}(d_4)$, i.e. the third homology $H_3(L_P)$ contains the image $\psi(x)$.

2.2. Poset without $\hat{0}$ or $\hat{1}$

When the poset P does not have $\hat{0}$ or $\hat{1}$, the r-chain is usually defined to be simply

$$x_1 < x_2 < \cdots < x_r,$$

and the boundary map is:

$$\partial_r(x_1 < \cdots < x_r) = \sum_{i=1}^{r} (-1)^{i-1}(x_1 < \cdots < \widehat{x_i} < \cdots < x_r).$$

In that case the insertion map will not do its job. For example, let $P = \{1 < 3, 2 < 3, 3 < 4\}$ be the poset represented by Hasse diagram in Fig. 2.

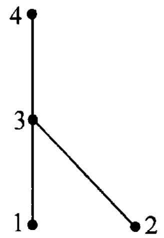

Fig. 2. Example: poset P.

Let $x = 1 < 3 < 4$, so $y = \psi(x) = z_{1,3} \wedge z_{3,4}$. Then

$$\psi(\delta(x)) = \psi(3 < 4 - 1 < 4 + 1 < 3) = z_{3,4} - z_{1,4} + z_{1,3}.$$

On the other hand:

$$\delta(\psi(x)) = \delta(z_{1,3} \wedge z_{3,4}) = z_{1,4} \neq \psi(\delta(x)).$$

The problem lies in the discrepancy of the definitions of the boundary maps for the poset P without $\hat{0}$ and $\hat{1}$ and the Lie algebra corresponding to that poset.

3. What next?

It would be interesting to expand this result to the posets without $\hat{0}$ and $\hat{1}$, and see if similar analog exists when the homology is evaluated over a field of polynomials up to the kth degree.

Also, the poset homology part of the Lie algebra homology is extremely small. How could we characterize the rest of the Lie algebra homology, in relation to the poset homology?

These questions and many others are worth investigating.

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